

# Lane-Emden Equation: perturbation method

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## ABSTRACT

The perturbation method is applied to numerical solution of the Lane-Emden Equation (LEE) of arbitrary index  $n$ , and the global parameters of polytropes are found as function of polytropic index  $n$ .

## 1. Introduction

The basic differential equation of internal structure of stars, the Lane-Emden equation (hereafter LEE) of index  $n$ , is solved analytically only for three values of  $n = 0, 1, 5$ . For other values of  $n$ , LEE can be solved only numerically. Recently, using the perturbation method, SK (Seidov & Kuzakhmedov, 1978 (SK78)) had presented the new *analytical* solutions of the LEE for index  $n$  only slightly differing from 0, 1, and 5, see also Seidov (1978a,b, 1979a,b, 2004); Jabbar (1984); Caimmi (1987); Horedt (1987, 1990); Medvedev & Rybicki (2001). In this paper I present the *numerical* perturbation method for solving the LEE, see also Seidov (2004).

## 2. Basic equation

The basic equation is LEE of index  $n$  :

$$LE = \frac{1}{x^2} \frac{d}{dx} \left( x^2 \frac{d}{dx} \right); \quad LE[y] = -y^n; \quad y(0) = 1, \quad y'(0) = 0, \quad (1)$$

where we introduced the Lane-Emden differential operator  $LE$ .

We look for solution  $y(x)$  in the interval  $[0, X]$  such that  $y(X) = 0$ .

Three classical analytical solutions of equation (1) are (see e.g. (Chandrasekhar 1957)):

$$n = 0, \quad y = 1 - \frac{1}{6} x^2, \quad X = \sqrt{6}, \quad \mu = 2\sqrt{6}, \quad \rho_c/\rho_m = 1; \quad (2)$$

$$n = 1, \quad y = \frac{\sin x}{x}, \quad X = \pi, \quad \mu = \pi, \quad \rho_c/\rho_m = \pi^2/3; \quad (3)$$

$$n = 5, \quad y = (1 + \frac{1}{3}x^2)^{-1/2}, \quad X \rightarrow \infty, \quad \mu = \sqrt{3}, \quad \rho_c/\rho_m \rightarrow \infty. \quad (4)$$

In these equations,  $\mu = -X^2 y'(X)$ ,  $\rho_c/\rho_m = X^3/3\mu$ ;  $X$ ,  $\mu$  are dimensionless radius and mass, and  $\rho_c/\rho_m$  is the central-to-mean density ratio.

### 3. The perturbation method

Consider equation (1) as ODE depending on parameter  $n$ , then assuming  $n = n_0 + \delta$  with  $\delta$  as a small parameter,  $\delta \ll n_0$ , (or  $\delta \ll 1$ , if  $n_0 = 0$ ) we expand the r.s. of equation (1) to the second order of  $\delta$  (for the sake of brevity we omit index at  $n$ ):

$$\begin{aligned} y &= y_0 + \delta y_1 + \delta^2 y_2; & y_0^{n+\delta} &= h_0 + \delta h_0 + \delta^2 h_1; \\ & & h_0 &= y_0^n; & h_1 &= n y_0^{n-1} y_1 + y_0^n \ln y_0; \\ h_2 &= \frac{1}{2} n(n-1) y_0^{n-2} y_1^2 + y_0^{n-1} (y_1 + n y_2 + n y_1 \ln y_0) + \frac{1}{2} y_0^n \ln^2 y_0. \end{aligned} \quad (5)$$

From equations (1,5) we have three coupled ODEs for three functions  $y_0$ ,  $y_1$ ,  $y_2$ :

$$LE[y_0] = -h_0, \quad y_0(0) = 1, \quad y_0'(0) = 0; \quad (6)$$

$$LE[y_1] = -h_1, \quad y_1(0) = y_1'(0) = 0; \quad (7)$$

$$LE[y_2] = -h_2, \quad y_2(0) = y_2'(0) = 0. \quad (8)$$

Initial conditions in equations (6,7,8) are defined by the form of series expansion of the solution of LEE of arbitrary  $n$  at  $x = 0$  (see further, formulas (16 - 22):

$$y = 1 - \frac{1}{6}x^2 + \frac{n}{5!}x^4 + \dots \quad (9)$$

Writing  $n = n_0 + \delta$ , expanding equation (9) to the second order of  $\delta$ , we have the series expansions for functions  $y_1$ ,  $y_2$  at  $x = 0$ :

$$y_1 = \frac{1}{5!}x^4 + \frac{5-16n}{3 \cdot 7!}x^6 \dots, \quad (10)$$

$$y_2 = -\frac{8}{3 \cdot 7!}x^6 + \frac{183(-1+2n)}{9 \cdot 9!}x^8 \dots \quad (11)$$

Note that the series expansion for  $y_1$  and  $y_2$  were given in Seidov (2004) only for case  $n = 0$ . Before solving eqs (6,7,8), I'd like to mention that the validity of the approach used is discussed briefly in SK78, and in Seidov (2004). For  $n = 0$ , 1, and 5 the *analytical* solutions

for functions  $y_1$  were presented in SK78; also the second approximation at the case of  $n = 0$  is partly given in Seidov (2004). Here we solve *numerically* equations (6 - 8). We remind that  $y_0$  is the "basic" solution of LEE and we refer to  $y_1$  and  $y_2$  as the "perturbed" LEE solutions of the first and second order. I widely used *MATEMATICA*'s (Wolfram 1999) function *NDSolve* with suitable options.

#### 4. System of ODEs

For a better numerical accuracy, we introduce three additional functions,  $z_0$ ,  $z_1$ , and  $z_2$ :

$$z_0 = -x^2 y_0', \quad z_1 = -x^2 y_1', \quad z_2 = -x^2 y_2', \quad (12)$$

and rewrite the three equations (6-8) as a system of six differential equations:

$$y_0' = -z_0/x^2, \quad z_0' = -x^2 h_0, \quad (13)$$

$$y_1' = -z_1/x^2, \quad z_1' = -x^2 h_1. \quad (14)$$

$$y_2' = -z_2/x^2, \quad z_2' = -x^2 h_2. \quad (15)$$

#### 5. Series solution of zero-th order

Again, for a larger numerical accuracy, we use the series solution of LEE. In Seidov & Kuzakhmedov (1977), the method of accurate series solution of LEE is given. Using the formulas of Seidov & Kuzakhmedov (1977), we present solution of LEE (13) at  $x = 0$  in the form

$$y_0 = 1 + \sum_{i=1}^{i=12} a_i x^{2i}, \quad z_0 = -2 \sum_{i=1}^{i=12} i a_i x^{2i+1}, \quad (16)$$

with coefficients  $a_i$  as follows:

$$a_1 = -\frac{1}{6}; \quad a_2 = \frac{n}{5!}; \quad a_3 = \frac{n(5-8n)}{3 \cdot 7!}; \quad a_4 = \frac{n(70-183n+122n^2)}{9 \cdot 9!};$$

$$a_5 = \frac{n(3150-10805n+12642n^2-5032n^3)}{45 \cdot 11!};$$

$$\begin{aligned}
a_6 &= \frac{n(138600 - 574850n + 915935n^2 - 663166n^3 + 183616n^4)}{135 \cdot 13!}; \\
a_7 &= n(21021000 - 101038350n + 199037015n^2 \\
&\quad - 200573786n^3 + 103178392n^4 - 21625216n^5)/945 \cdot 15!; \\
a_8 &= n(1891890000 - 10267435500n + 23780949500n^2 - 30057075285n^3 \\
&\quad + 21827357636n^4 - 8618115372n^5 + 1442431856n^6)/2835 \cdot 17!; \\
a_9 &= n(675404730000 - 4066235428500n + 10740122081500n^2 \\
&\quad - 16120795594195n^3 + 14830640277988n^4 - 8348507232868n^5 \\
&\quad + 2657923739344n^6 - 368552598784n^7)/25515 \cdot 19!; \\
a_{10} &= n(171102531600000 - 1128186384570000n \\
&\quad + 3329284073314500n^2 - 5740042719521900n^3 \\
&\quad + 6317195348852735n^4 - 4538114873629364n^5 \\
&\quad + 2074925918891156n^6 - 551199819782480n^7 \\
&\quad + 65035924972928n^8)/127575 \cdot 21!; \\
a_{11} &= n(118574054398800000 - 847953056599110000n \\
&\quad + 2754994980587692500n^2 - 5335484162711174500n^3 \\
&\quad + 6782008348777403475n^4 - 5860922969087284308n^5 \\
&\quad + 3438918097715059380n^6 - 1319254687791147504n^7 \\
&\quad + 299840088682556928n^8 - 30720693974199296n^9)/1403325 \cdot 23!; \\
a_{12} &= n(27272032511724000000 - 209899877314257900000n + 742585473289204545000n^2 \\
&\quad - 1589853990586539282500n^3 + 2279636465710370388750n^4 - 2285217511971127632065n^5 \\
&\quad + 1620103707989338077938n^6 - 801095938682391176900n^7 + 264081052577164986584n^8 \\
&\quad - 52342890902954850528n^9 + 4731477379473053696n^{10})/4209975 \cdot 25!. \quad (17)
\end{aligned}$$

We remind that the general recurrence relation for coefficient  $a_i$  is (Seidov & Kuzakhmedov (1977)):

$$a_{m+1} = \frac{1}{m(m+1)(2m+3)} \sum_{i=1}^{i=m} (in + i - m)(m+1-i)[3+2(m-i)]a_i a_{m+1-i}. \quad (18)$$

## 6. Series solution of the first order

Writing  $n = n_0 + \delta$ , expanding equation (17) to the *first* order of  $\delta$ , we have, from (16), the series expansions for solutions  $y_1, z_1$  of the equations (14) at  $x = 0$ :

$$y_1 = 1 + \sum_{i=2}^{i=12} b_i x^{2i}, \quad z_1 = -2 \sum_{i=2}^{i=12} i b_i x^{2i+1}, \quad (19)$$

with coefficients  $b_i$  as follows:

$$\begin{aligned} b_2 &= 1/5!; \quad b_3 = (5 - 16n)/3 \cdot 7!; \quad b_4 = (70 - 366n + 366n^2)/9 \cdot 9!; \\ b_5 &= (3150 - 21610n + 37926n^2 - 20128n^3)/45 \cdot 11!; \\ b_6 &= (138600 - 1149700n + 2747805n^2 - 2652664n^3 + 918080n^4)/135 \cdot 13!; \\ b_7 &= (21021000 - 202076700n + 597111045n^2 - 802295144n^3 \\ &\quad + 515891960n^4 - 129751296n^5)/945 \cdot 15!; \\ b_8 &= 4(472972500 - 5133717750n + 17835712125n^2 - 30057075285n^3 \\ &\quad + 27284197045n^4 - 12927173058n^5 + 2524255748n^6)/2835 \cdot 17!; \\ b_9 &= 4(168851182500 - 2033117714250n + 8055091561125n^2 \\ &\quad - 16120795594195n^3 + 18538300347485n^4 \\ &\quad - 12522760849302n^5 + 4651366543852n^6 - 737105197568n^7)/25515 \cdot 19!; \\ b_{10} &= (171102531600000 - 2256372769140000n + 9987852219943500n^2 \\ &\quad - 22960170878087600n^3 + 31585976744263675n^4 \\ &\quad - 27228689241776184n^5 + 14524481432238092n^6 \\ &\quad - 4409598558259840n^7 + 585323324756352n^8)/127575 \cdot 21!; \\ b_{11} &= (118574054398800000 - 1695906113198220000n \\ &\quad + 8264984941763077500n^2 - 21341936650844698000n^3 \\ &\quad + 33910041743887017375n^4 - 35165537814523705848n^5 \\ &\quad + 24072426684005415660n^6 - 10554037502329180032n^7 \\ &\quad + 2698560798143012352n^8 - 307206939741992960n^9)/1403325 \cdot 23!; \\ b_{12} &= 2(13636016255862000000 - 209899877314257900000n \\ &\quad + 1113878209933806817500n^2 - 3179707981173078565000n^3 \\ &\quad + 5699091164275925971875n^4 - 6855652535913382896195n^5 \\ &\quad + 5670362977962683272783n^6 - 3204383754729564707600n^7 \\ &\quad + 1188364736597242439628n^8 - 261714454514774252640n^9 \\ &\quad + 26023125587101795328n^{10})/4209975 \cdot 25!. \end{aligned} \quad (20)$$

## 7. Series solution of the second order

Again, as in section 6, writing  $n = n_0 + \delta$ , expanding equation (17) to the *second* order of  $\delta$ , we have get the series expansions for solutions  $y_2, z_2$  of the equations (15) at  $x = 0$ :

$$y_2 = 1 + \sum_{i=3}^{i=12} c_i x^{2i}, \quad z_2 = -2 \sum_{i=3}^{i=12} i c_i x^{2i+1}, \quad (21)$$

with coefficients  $c_i$  as follows:

$$\begin{aligned} c_3 &= -1/3 \cdot 7!; \quad c_4 = 61(-1 + 2n)/3 \cdot 9!; \\ c_5 &= (-10805 + 37926n - 30192n^2)/45 \cdot 11!; \\ c_6 &= (-574850 + 2747805n - 3978996n^2 + 1836160n^3)/135 \cdot 13!; \\ c_7 &= (-101038350 + 597111045n - 1203442716n^2 \\ &\quad + 1031783920n^3 - 324378240n^4)/945 \cdot 15!; \\ c_8 &= 2(-5133717750 + 35671424250n - 90171225855n^2 \\ &\quad + 109136788180n^3 - 64635865290n^4 + 15145534488n^5)/2835 \cdot 17!; \\ c_9 &= 2(-2033117714250 + 16110183122250n - 48362386782585n^2 \\ &\quad + 74153201389940n^3 - 62613804246510n^4 \\ &\quad + 27908199263112n^5 - 5159736382976n^6)/25515 \cdot 19!; \\ c_{10} &= 2(-564093192285000 + 4993926109971750n \\ &\quad - 17220128158565700n^2 + 31585976744263675n^3 \\ &\quad - 34035861552220230n^4 + 21786722148357138n^5 \\ &\quad - 7716797476954720n^6 + 1170646649512704n^7)/127575 \cdot 21!; \\ c_{11} &= 2(-141325509433185000 + 1377497490293846250n \\ &\quad - 5335484162711174500n^2 + 11303347247962339125n^3 \\ &\quad - 14652307422718210770n^4 + 12036213342002707830n^5 \\ &\quad - 6156521876358688352n^6 + 1799040532095341568n^7 \\ &\quad - 230405204806494720n^8)/467775 \cdot 23!; \\ c_{12} &= (-209899877314257900000 + 2227756419867613635000n \\ &\quad - 9539123943519235695000n^2 + 22796364657103703887500n^3 \\ &\quad - 34278262679566914480975n^4 + 34022177867776099636698n^5 \\ &\quad - 22430686283106952953200n^6 + 9506917892777939517024n^7 \\ &\quad - 2355430090632968273760n^8 + 260231255871017953280n^9)/4209975 \cdot 25!. \end{aligned} \quad (22)$$

## 8. Cases with analytical solutions

We first consider three cases, for which the zero-th and first approximations have exact analytical solutions.

### 8.1. n=5 case

We first mention that in this case coefficients  $a_i$  can be found from the very simple recurrence relation:

$$n = 5 : \quad a_m = \frac{1}{3m} \left( \frac{3}{2} - m \right) a_{m-1}, \quad m \geq 1, \quad a_0 = 1, \quad (23)$$

which can be used e.g. for checking formulas (16, 17). Then, we calculate, for  $n = 5$ , values of  $y_0$  and  $z_0$  according to (16) and (17) at point  $x_i = 1/100$ . With using *MATHEMATICA* it can be done *exactly* (without any *numerical error*). Then we compare these values with exact analytical values of  $y_0$  and  $z_0$  at  $x_i = 1/100$  (see (4)):

$$\begin{aligned} \frac{y_0(x_i)_{15,16}}{(1 + x_i^2/3)^{-1/2}} - 1 &= 9.72065376 \cdot 10^{-60}, \\ \frac{z_0(x_i)_{15,16}}{(x_i^3/3)(1 + x_i^2/3)^{-3/2}} - 1 &= -2.5273216 \cdot 10^{-60}. \end{aligned} \quad (24)$$

Hence we have some 60 digits of accuracy in the initial values for LEE (13). Now we solve the system (13-15) numerically using *NDSolve* procedure of *MATHEMATICA*, from  $x = x_i = 1/100$  till  $x = x_f = 10$  with options: Method  $\rightarrow$  RungeKutta, AccuracyGoal  $\rightarrow$  Infinity, PrecisionGoal  $\rightarrow$  32, WorkingPrecision  $\rightarrow$  24, MaxSteps  $\rightarrow$  50000, and compare

the numerical values with values from analytical expressions when available

$n = 5$	$x_f = 10$	
$y_0(x_f), \text{NDSolve}$	0.17066 40371 96572 26797 786	
$y_0(x_f), \text{Analytic}$	0.17066 40371 96572 28860 143	
$z_0(x_f), \text{NDSolve}$	1.65693 23999 66721 38805 555	
$z_0(x_f), \text{Analytic}$	1.65693 23999 66721 24855 754	
$y_1(x_f), \text{NDSolve}$	0.096926 10440 49489 40514 448	(25)
$y_1(x_f), \text{Analytic}$	0.096926 10440 49489 36327 188	
$z_1(x_f), \text{NDSolve}$	-0.12538 65905 38367 83413 8922	
$z_1(x_f), \text{Analytic}$	-0.12538 65905 38367 81528 4516	
$y_2(x_f), \text{NDSolve}$	-0.0131 49071 88483 68785	
$z_2(x_f), \text{NDSolve}$	0.0001 04233 84608 82482.	

Final results are of 15-16 digits accuracy. Here "analytical" values for  $y_1$  and  $z_1$  are given according to analytical solution SK78:

$$n = 5 : \quad \nu = \arctan\left(\frac{x}{\sqrt{3}}\right),$$

$$y_1 = \frac{1}{48 \sin(\nu)} \left( \sin(2\nu) - \frac{5}{4} \sin(4\nu) + 3 \nu \cos(4\nu) - 3 [2 \sin(2\nu) + \sin(4\nu)] \ln[\cos(\nu)] \right),$$

$$z_1 = -x^2 \frac{dy_1}{dx}. \quad (26)$$

## 8.2. n=1 case

In the case of  $n = 1$  we first mention that coefficients  $a_i$  are of very simple form:

$$n = 1 : \quad a_i = \frac{1}{(2i+1)!}, \quad (27)$$

which can be used e.g. for checking formulas (16, 17). Then, we calculate, for  $n = 1$ , values of  $y_0$  and  $z_0$  according to (16) and (17) at point  $x_i = 1/100$ , and compare these values with

exact analytical values of  $y_0$  and  $z_0$  at  $x_i = 1/100$ :

$$\begin{aligned} \frac{y_0(x_i)_{16,17}}{\sin(x_i)/x_i} - 1 &= 9.183841796 \cdot 10^{-81}, \\ \frac{z_0(x_i)_{16,17}}{\sin(x_i) - x_i \cos(x_i)} - 1 &= -2.3877590 \cdot 10^{-81}. \end{aligned} \quad (28)$$

Now we have some 80 digits of accuracy in the initial values for LEE (13). Then we solve the system (13 - 15) numerically using *NDSolve* procedure of *MATHEMATICA*, from  $x = x_i = 1/100$  till  $x = x_f = \pi$  with the same options as for case  $n = 5$ , section 8.1, and compare the numerical values with values from analytical expressions when available

$n = 1$	$X = \pi$	
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$y_0(X), \text{NDSolve}$	$-6.552265490 \cdot 10^{-19}$	
$y_0(X), \text{Analytic}$	0	
-----	-----	
$z_0(X), \text{NDSolve}$	3.14159 26535 89793 22713 584	
$z_0(X), \text{Analytic}, \pi$	3.14159 26535 89793 23846 264	
-----	-----	
$y_1(X), \text{NDSolve}$	0.28179 14499 02207 82015 93981	
$y_1(X), \text{Analytic}$		
$\frac{1}{4\pi} \text{Si}(2\pi) + \frac{1}{2} \ln(2\pi) - \frac{3}{4}$	0.28179 14499 02207 82012 97922	(29)
-----	-----	
$z_1(X), \text{NDSolve}$	$-1.02925 45490 49509 89738 663$	
$z_1(X), \text{Analytic}$		
$\frac{1}{4} \text{Si}(2\pi) + \frac{\pi}{4} [\text{Ci}(2\pi) + \ln(2\pi) - 3 - \text{EulerGamma}]$	$-1.02925 45490 49509 87940 655$	
-----	-----	
$y_2(X), \text{NDSolve}$	$-0.09455 03188 95873 420$	
$y_2(X), \text{Analytic}$	—	
-----	-----	
$z_2(X), \text{NDSolve}$	$-0.81171 80531 86985 23026$	
$z_2(X), \text{Analytic}$	—	
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Again, as  $n = 5$  case, final results are of 15-16 digits accuracy, and though there is no guarantee that the similar accuracy will be in other numerical results, still we will use the same set of options for solving other cases of  $n$ .

### 8.3. n=0 case

In the case of  $n = 0$  we first mention that coefficients  $a_i$  are of very simple form:

$$n = 0 : \quad a_1 = -\frac{1}{6}, \quad a_{i>1} = 0; \quad (30)$$

while coefficients  $b_i$  can be found from the series expansion of the known function  $y_1$  (SK78):

$$n = 0 : \quad y_1(x) = \frac{5x^2}{18} - 4 + (2 + y_0) \ln(y_0) + 4 \frac{\sqrt{6}}{x} \operatorname{arctanh}\left(\frac{x}{\sqrt{6}}\right), \quad (31)$$

with (see equation (2)):

$$n = 0 : \quad y_0 = 1 - \frac{1}{6}x^2,$$

and

$$y_1(\sqrt{6}) = 4 \ln 2 - \frac{7}{3}. \quad (32)$$

Then we solve, for  $n = 0$ , the system (13-15) numerically using *NDSolve* procedure of *MATHEMATICA*, from  $x = x_i = 1/100$  till  $x = X = \sqrt{6}$  with the same options as for cases  $n = 5, 1$ , and compare the numerical values with values from analytical expressions when

available:

$n = 0$	$X = \sqrt{6}$
$y_0(X), \text{NDSolve}$	$1.530418314373 \cdot 10^{-31}$
$y_0(X), \text{Analytic}$	$0$
$z_0(X), \text{NDSolve}$	$4.89897\,94855\,66356\,19639\,45681\,49410\,81340$
$z_0(X), \text{Analytic} = 2\sqrt{6}$	$4.89897\,94855\,66356\,19639\,45681\,49411\,78278$
$y_1(X), \text{NDSolve}$	$0.43925\,53889\,06447\,90287$
$y_1(X), \text{Analytic} = 4 \ln 2 - \frac{7}{3}$	$0.43925\,53889\,06447\,90433$
$z_1(X), \text{NDSolve}$	$-6.27251\,76587\,60946\,466$
$z_1(X), \text{Analytic} = \frac{4}{3}\sqrt{6} (3 \ln 2 - 4)$	$-6.27251\,76587\,60954\,355$
$y_2(X), \text{NDSolve}$	$-0.41351\,38937\,125$
$y_2(X), \text{Analytic} =$ $= \frac{1}{9}(413 - 21 \pi^2 - 402 \ln 2 + 144 \ln^2 2)$	$-0.41351\,38893\,059$
$z_2(X), \text{NDSolve}$	$-2.59850 \cdot 10^{12}$
$z_2(X), \text{Analytic}$	$\infty$

(33)

We see that numerical accuracy is drastically decreasing for the larger degrees of perturbation: while  $y_0(X)$ , and  $z_0(X)$  are calculated with some 30-31 correct digits,  $y_1(X)$ , and  $z_1(X)$  are calculated with 15-16 correct digits, and  $y_2(X)$ , and  $z_2(X)$  only with 8-9 correct digits.

## 9. Non-analytical cases

For other values of  $n$ , there is no analytic solution even for  $y_0$  and there is no possibility to check numerical results.

Moreover we should calculate numerically LEE to find accurate values of the first zero  $X$ . Also we give comparison of our numerical values of  $X$  and  $z_0(X)$  with two most accurate results known in literature (Jabbar (1984), Horedt (1990)).

**9.1. n=1/2 case**

$n = 1/2$		
$X$ , NDSolve	2.75269 80540 64991	
$X$ , Jabbar (1984)	2.75269 80541	
$X$ , Horedt (1990)	2.75269 8	
$y_0(X)$ , NDSolve	$-1.57415358 \cdot 10^{-15}$	
$y_0(X)$ , Analytic	0	
$z_0(X)$ , NDSolve	3.78865 11848 84005 74354 209	(34)
$z_0(X)$ , Horedt (1990)	3.78865 11652 97791	
$y_1(X)$ , NDSolve	0.34123 20401 31647 34451	
$z_1(X)$ , NDSolve	$-1.62970 77128 59769 520$	
$y_2(X)$ , NDSolve	$-0.14682 13064 84602 959$	
$z_2(X)$ , NDSolve	$-3.71857 97123 08979 172$	

### 9.2. n=3/2 case

$n = 3/2$	
$X$ , NDSolve	3.65375 37362 19119
$X$ , Jabbar (1984)	3.65375 37362
$X$ , Horedt (1990)	3.65375 4
$y_0(X)$ , NDSolve	$6.8926652958 \cdot 10^{-16}$
$y_0(X)$ , Analytic	0
$z_0(X)$ , NDSolve	2.71405 51201 08646
$z_0(X)$ , Horedt (1990)	2.71405 57437
$y_1(X)$ , NDSolve	0.24067 09191 40952 32
$z_1(X)$ , NDSolve	−0.71009 78273 59729
$y_2(X)$ , NDSolve	−0.06656 18185 10763 63
$z_2(X)$ , NDSolve	0.24186 12563 97462 12

(35)

### 9.3. n=2 case

In this case we first mention that recurrence relation for coefficients  $a_i$ , may be obtained in the more simple form than in (17). This relation, according to Seidov (1979a), is:

$$a_{i+1} = -\frac{1}{(2i+2)(2i+3)} \sum_{k=0}^{k=i} a_k a_{i-k}, \quad i \geq 0, \quad a_0 = 1. \quad (36)$$

From equation (36) it follows that the series is sign-alternating, the result not evident from (17). In general, it can be shown, that coefficients  $a_i$  of serial solution for LEE of *integer* index  $n$  is sign-alternating.

We present results of numerical solution of the system (13-15).

$n = 2$		
$X$ , NDSolve	4.35287 45959 46124	
$X$ , Jabbar (1984)	4.35287 45959	
$X$ , Horedt (1990)	4.35287 5	
$y_0(X)$ , NDSolve	$3.891892651074 \cdot 10^{-17}$	
$y_0(X)$ , Analytic	0	
$z_0(X)$ , NDSolve	2.41104 60120 96894	(37)
$z_0(X)$ , Horedt (1990)	2.41104 73856	
$y_1(X)$ , NDSolve	0.20990106855590432	
$z_1(X)$ , NDSolve	$-0.51626996767525199$	
$y_2(X)$ , NDSolve	$-0.049552916011399585$	
$z_2(X)$ , NDSolve	0.15512703899482025	

#### 9.4. $n=5/2$ case

We present results of numerical solution of the system (13-15).

$n = 5/2$		
$X$ , NDSolve	5.35527 54590 10824	
$X$ , Jabbar (1984)	5.35527 54459 0	
$X$ , Horedt (1990)	5.35527 5	
$y_0(X)$ , NDSolve	$-3.385331 \cdot 10^{-15}$	
$y_0(X)$ , Analytic	0	
$z_0(X)$ , NDSolve	2.18719 95655 17079	(38)
$z_0(X)$ , Horedt (1990)	2.18719 90907	
$y_1(X)$ , NDSolve	0.18557608273853998	
$z_1(X)$ , NDSolve	$-0.38690411187876034$	
$y_2(X)$ , NDSolve	$-0.03834028384550369$	
$z_2(X)$ , NDSolve	0.10810493285592643	

**9.5. n=3 case**

$n = 3$		
$X$ , NDSolve	6.89684 86193 7696	
$X$ , Jabbar (1984)	6.89684 86194	
$X$ , Horedt (1990)	6.89684 9	
$y_0(X)$ , NDSolve	$1.7861333071518752 \cdot 10^{-17}$	
$y_0(X)$ , Analytic	0	
$z_0(X)$ , NDSolve	2.01823 59509 66228 7	(39)
$z_0(X)$ , Horedt (1990)	2.01823 62876	
$y_1(X)$ , NDSolve	0.16547 68294 39454 83	
$z_1(X)$ , NDSolve	−0.29335 99255 39480 26	
$y_2(X)$ , NDSolve	−0.03053 51512 78657 177	
$z_2(X)$ , NDSolve	0.08152 75811 87124 82	

**9.6.  $n=7/2$  case**

$n = 7/2$		
$X$ , NDSolve	9.53580 53442 4479	
$X$ , Jabbar (1984)	9.53580 53443	
$X$ , Horedt (1990)	9.535805	
$y_0(X)$ , NDSolve	$1.2322310495888572 \cdot 10^{-15}$	
$y_0(X)$ , Analytic	0	
$z_0(X)$ , NDSolve	1.89055 70934 43116 4	(40)
$z_0(X)$ , Horedt (1990)	1.89055 65987	
$y_1(X)$ , NDSolve	0.1481615824132385	
$z_1(X)$ , NDSolve	−0.21964831524439676	
$y_2(X)$ , NDSolve	−0.024934678004649848	
$z_2(X)$ , NDSolve	0.06771308534193725	

**9.7. n=4 case**

$n = 4$		
$X$ , NDSolve	14.97154 63488 38093	
$X$ , Jabbar (1984)	14.97154 63496	
$X$ , Horedt (1990)	14.97155	
$y_0(X)$ , NDSolve	$9.088953198578 \cdot 10^{-18}$	
$y_0(X)$ , Analytic	0	
$z_0(X)$ , NDSolve	1.79722 99144 3925	
$z_0(X)$ , Horedt (1990)	1.79723 08344	(41)
$y_0''(X)$ , NDSolve	0.00107 11089 71589 365	
$y_1(X)$ , NDSolve	0.13250 00877 72273 87	
$z_1(X)$ , NDSolve	−0.15399 30629 85788 74	
$y_2(X)$ , NDSolve	−0.02097 11409 23665 716	
$z_2(X)$ , NDSolve	0.06552 46752 25231 59	

### 9.8. n=9/2 case

$n = 9/2$	
$X$ , NDSolve	31.83646 32446 95228
$X$ , Jabbar (1984)	31.83646 32485
$X$ , Horedt (1990)	31.83646
$y_0(X)$ , NDSolve	$-1.6287136623021175 \cdot 10^{-15}$
$y_0(X)$ , Analytic	0
$z_0(X)$ , NDSolve	1.73779 88676 66032 3
$z_0(X)$ , Horedt (1990)	1.73779 86022
$y_1(X)$ , NDSolve	0.11719722297815073
$z_1(X)$ , NDSolve	-0.07993661318489896
$y_2(X)$ , NDSolve	-0.018685417777349604
$z_2(X)$ , NDSolve	0.08780938436898907

(42)

## 10. Global parameters

### 10.1. Radius

Radius of the polytrope is proportional to  $X$ , the first zero of  $y(x)$ . In our second order approximation in  $\delta$ , we write:

$$X = X_0 + \delta X_1 + \delta^2 X_2, \quad y(X) = 0, \quad y(X) = y_0(X) + \delta y_1(X) + \delta^2 y_2(X), \quad (43)$$

from where we get:

$$X_1 = \frac{y_1(X_0)}{-y'_0(X_0)}; \quad X_2 = \frac{y_2(X_0) + X_1 y'_1(X_0) + 1/2 X_1^2 y''_0(X_0)}{-y'_0(X_0)}. \quad (44)$$

Let me remind the meaning of  $X_0$ ,  $X_1$ , and  $X_2$ : for each  $n$ ,  $X_0$  is the dimensional radius, while  $X_1$ , and  $X_2$  define the derivatives of  $X(n)$  by  $n$ :  $X_0 = X(n)$ ,  $X_1 = dX(n)/dn$ , and  $2X_2 = d^2X(n)/dn^2$ .

## 10.2. Mass

Mass of the polytrope is proportional to  $\mu = -X^2 y'(X)$ . In our second order approximation in  $\delta$ , we write:

$$\mu = \mu_0 + \delta \mu_1 + \delta^2 \mu_2, \quad (45)$$

from where we get:

$$\mu_0 = -X_0^2 y'_0(X_0), \quad \mu_1 = -X_0^2 [X_1 y''_0(X_0) + y'_1(X_0)] - 2X_0 X_1 y'_0(X_0). \quad (46)$$

$$\begin{aligned} \mu_2 = & -X_0^2 [X_2 y''_0(X_0) + \frac{1}{2} X_1^2 y'''_0(X_0) + X_1 y''_1(X_0) + y'_2(X_0)] \\ & - 2X_0 X_1 (X_1 y''_0(X_0) + y'_1(X_0)) - y'_0(X_0) (X_1^2 + 2X_0 X_2). \end{aligned} \quad (47)$$

Let me remind the meaning of  $\mu_0$ ,  $\mu_1$ , and  $\mu_2$ : for each  $n$ ,  $\mu_0$  is the dimensional mass, while  $\mu_1$ , and  $\mu_2$  define the derivatives of  $\mu(n)$  by  $n$ :  $\mu_0 = \mu(n)$ ,  $\mu_1 = d\mu(n)/dn$ , and  $2\mu_2 = d^2\mu(n)/dn^2$ .

10.2.1.  $n=0$

$$\begin{aligned} n &= 0, \\ X_0 &= \sqrt{6}, \\ y_0'(X_0) &= -0.81649\,65809\,277260, \\ y_0''(X_0) &= -0.33333\,33333\,33333, \\ y_0'''(X_0) &= 0, \\ y_1(X_0) &= 0.43925\,53889\,064479, \\ y_1'(X_0) &= 1.04541\,96097\,934910777, \\ y_1''(X_0) &= 70.10101\,52396\,6198, \\ y_2(X_0) &= -0.41351\,38937\,12500\,52051\,096, \\ y_2'(X_0) &= 4.33084\,54146\,19353 \cdot 10^{11}, \\ X_1 &= 0.53797\,57847\,94289\,96\,7933774, \\ X_2 &= 0.12328\,30846\,88831\,89\,0159441, \\ X(n = .1) &= 2.50452\,01521\,09495, (*2.5045449630\,Jabbar*) \\ X(n = -.1) &= -, \\ \mu_0 &= 4.89897\,94855\,66356\,196, \\ \mu_1 &= -3.04466\,29499\,952067, \\ \mu_2 &= -2.59850\,72489\,991943 \cdot 10^{12}, \\ \mu(n = .1) &= 4.59451\,31905, (*Horedt*) \\ \mu(n = -.1) &= -. \end{aligned} \tag{48}$$

10.2.2.  $n=1/2$

$$\begin{aligned} n &= 1/2, \\ X_0 &= 2.75269805406499, \\ y_0'(X_0) &= -0.49999708294422937513122874, \\ y_0''(X_0) &= 0.3632777637699180575486, \\ y_0'''(X_0) &= 3.7771115383 \cdot 10^6, \\ y_1(X_0) &= 0.34123204013164497867521625893, \\ y_1'(X_0) &= 0.215076348248895845791, \\ y_1''(X_0) &= -2.723045849967156701 \cdot 10^6, \\ y_2(X_0) &= -0.146821475444243995570334, \\ y_2'(X_0) &= 929115.93812128759, \\ X_1 &= 0.6824680618580861779508, \\ X_2 &= 0.169124728807637034976, \\ X(n = .6) &= 2.822636107538865, (*2.8226750739, Jabbar*) \\ X(n = .4) &= 2.686142495167248, (*2.6861053263, Jabbar*) \\ \mu_0 &= 3.7886511848840057435396, \\ \mu_1 &= -1.6297077071514047746998, \\ \mu_2 &= 0, \\ \mu(n = .6) &= 3.6256804141688654, (\text{only 1st appr!}) \\ (*0.4560739 * 2.822675^2 &= 3.63376\,61327, \text{Horedt}) \\ \mu(n = .4) &= 3.9516219555991463, (\text{only 1st appr!}) \\ (*0.5489336 * 2.686105^2 &= 3.96064\,37923, \text{Horedt}). \end{aligned} \tag{49}$$

10.2.3.  $n=1$

$$\begin{aligned} X_0 &= \pi, \\ y_0'(X_0) &= -0.31830988618379067039, \\ y_0''(X_0) &= 0.202642367284675542812372, \\ y_0'''(X_0) &= 0.12480067956938'3.2195, \\ y_1(X_0) &= 0.2817914499022078201593, \\ y_1'(X_0) &= 0.104285289178956941658, \\ y_1''(X_0) &= -0.348181526960602809649062, \\ y_2(X_0) &= -0.09455031889587342081743776, \\ y_2'(X_0) &= 0.0822442339327594290586, \\ X_1 &= 0.88527394885719235129261239, \\ X_2 &= 0.2424591740663775292855118679, \\ X(n = 1.1) &= 3.232544640216176, (* 3.2326084072, Jabbar*) \\ X(n = 0.9) &= 3.0554898504447374, (* 3.0554293447, Jabbar*) \\ \mu_0 &= 3.1415926535897932271, \\ \mu_1 &= -1.02925454904951, \\ \mu_2 &= 0.4193306804206245, \\ \mu(n = 1.1) &= 3.0428605054890485, \\ (*0.2911738 * 3.232608^2 &= 3.042694721493137, Horedt*) \\ \mu(n = .9) &= 3.2487114152989505, \\ (*0.3055429 * 3.055429^2 &= 3.24889082611291, Horedt*). \end{aligned} \tag{50}$$

10.2.4.  $n=3/2$

$$\begin{aligned} n &= 3/2, \\ X_0 &= 3.653753736219119, \\ y_0'(X_0) &= -0.203301282638546737276897, \\ y_0''(X_0) &= 0.111283516797123609328, \\ y_0'''(X_0) &= -0.09137193541041363, \\ y_1(X_0) &= 0.24067091914095230177155721, \\ y_1'(X_0) &= 0.0531911817234202858, \\ y_1''(X_0) &= -0.02911591910998444970746, \\ y_2(X_0) &= -0.066561818510763637968669, \\ y_2'(X_0) &= -0.018117061781318394264, \\ X_1 &= 1.183814071497255432331730, \\ X_2 &= 0.3658800713499040420899, \\ X(n=1.6) &= 3.7757939440823436, (* 3.77590 47640, Jabbar*) \\ X(n=1.4) &= 3.5390311297828925, (* 3. (* 3.53892 66160, Jabbar*) \\ \mu_0 &= 2.7140551201086457355, \\ \mu_1 &= -0.71009782735972906926, \\ \mu_2 &= 0.24186141001555074558, \\ \mu(n=1.6) &= 2.64546 39514 728284, \\ (* 0.185544 * 3.775905^2 &= 2.64538 58927, Horedt*) \mu(n=1.4) = 2.78748 35169 44774, \\ (* 0.2225779 * 3.538927^2 &= 2.78756 657920, Horedt*). \end{aligned} \tag{51}$$

10.2.5.  $n=2$

$$\begin{aligned} n &= 2, \\ X_0 &= 4.352874595946124, \\ y_0'(X_0) &= -0.12724865113117523, \\ y_0''(X_0) &= 0.05846649074139796, \\ y_0'''(X_0) &= -0.040295089683388775, \\ y_1(X_0) &= 0.20990106855590432, \\ y_1'(X_0) &= 0.027247367605845233, \\ y_1''(X_0) &= -0.012519252280422236, y_2(X_0) = -0.049552916011399585, \\ y_2'(X_0) &= -0.008187196082955055, \\ X_1 &= 1.6495347234724416, \\ X_2 &= 0.5888879298143976, \\ X(n=2.1) &= 4.523716947591513, (* 4.5239262993, Jabbar *) \\ X(n=1.9) &= 4.193810002897024, (* 4.1936146217, Jabbar *) \\ \mu_0 &= 2.411046012096894, \\ \mu_1 &= -0.5162699676752518, \\ \mu_2 &= 0.15512703899481983, \\ \mu(n=2.1) &= 2.360970285719317, \\ (* 0.1153592 * 4.523926^2 &= 2.3609305957478286, Horedt *) \\ \mu(n=1.9) &= 2.4642242792543674, \\ (* 0.140123 * 4.193615^2 &= 2.4642600755839923, Horedt *). \end{aligned} \tag{52}$$

10.2.6.  $n=5/2$

$$\begin{aligned} n &= 5/2, \\ X_0 &= 5.3552754590108239, \\ y_0'(X_0) &= -0.076264913479991799518, \\ y_0''(X_0) &= 0.0284821627061845798710092, \\ y_0'''(X_0) &= -0.0159555729247840799988, \\ y_1(X_0) &= 0.1855760827385399754972, \\ y_1'(X_0) &= 0.01349086250870339952973971, \\ y_1''(X_0) &= -0.005038344941156511, \\ y_2(X_0) &= -0.03834028384550369027000, \\ y_2'(X_0) &= -0.003769483809799692567, \\ X_1 &= 2.43330\ 87690\ 08517\ 98883, \\ X_2 &= 1.03335\ 16315\ 07183\ 66045, \\ X(n=2.6) &= 5.608939852226747, (*\ 5.60938\ 27386, \text{Jabbar} *) \\ X(n=2.4) &= 5.122278098425044, (*\ 5.122278, \text{Jabbar} *) \\ \mu_0 &= 2.18719956551707901061, \\ \mu_1 &= -0.3869041118787603560, \\ \mu_2 &= 0.1081049328559264189, \\ \mu(n=2.6) &= 2.14959\ 02036\ 57762, \\ (*\ 0.06831578 * 5.609382^2 &= 2.14956\ 73869, \text{Horedt} *) \\ \mu(n=2.4) &= 2.22697\ 10260\ 33514, \\ (*\ 0.08489109 * 5.121870^2 &= 2.22699\ 48490, \text{Horedt} *). \end{aligned} \tag{53}$$

10.2.7.  $n=3$

$$\begin{aligned} n &= 3, \\ X_0 &= 6.89684861937696, \\ y_0'(X_0) &= -0.04242975760445296, \\ y_0''(X_0) &= 0.012304100016127638, \\ y_0'''(X_0) &= -0.005352053102589007, \\ y_1(X_0) &= 0.16547682943945483, \\ y_1'(X_0) &= 0.006167361415567608, \\ y_1''(X_0) &= -0.0017884578177459688, \\ y_2(X_0) &= -0.030535151278657177, \\ y_2'(X_0) &= -0.0017139698191338721, \\ X_1 &= 3.9000182603467954, \\ X_2 &= 2.0525978144309858, \\ X(n=3.1) &= 7.30737\,64235, (*7.30848\,42924\,\text{Jabbar}*) \\ X(n=2.9) &= ", 6.52737277148659, (*6.52637\,41261\,\text{Jabbar}*) \\ \mu_0 &= 2.0182359509662287, \\ \mu_1 &= ", -0.29335992553948076, \\ \mu_2 &= 0.08152758134658877, \\ \mu(n=3.1) &= ", 1.98971\,52342\,25746, \\ &(*0.03725063 * 7.308484^2 = 1.98970\,28553, \text{Horedt}*) \\ \mu(n=2.9) &= 2.0483872193336428, \\ &(*0.04809180 * 6.526374^2 = 2.04840\,08528, \text{Horedt}*). \end{aligned} \tag{54}$$

10.2.8.  $n=7/2$

$$\begin{aligned} n &= 7/2, \\ X_0 &= 9.53580534424479, \\ y_0'(X_0) &= -0.020790983939331436621, \\ y_0''(X_0) &= 0.00436061416708334168, \\ y_0'''(X_0) &= -0.00137186551413256, \\ y_1(X_0) &= 0.148161582413238524348, \\ y_1'(X_0) &= 0.0024155338182516865, \\ y_1''(X_0) &= -0.00050662397795474, \\ y_2(X_0) &= -0.0249346780046498475, \\ y_2'(X_0) &= -0.0007446596956576678, \\ X_1 &= 7.12624197323115573338729, \\ X_2 &= 4.9541783719117226, \\ X(n=3.6) &= 10.297971325287023, (* 10.30159 12890, Jabbar *) \\ X(n=3.4) &= 8.872722930640792, (* 8.8695802788, Jabbar *) \\ \mu_0 &= 1.890557093443116455, \\ \mu_1 &= -0.219648315244396768, \\ \mu_2 &= 0.067713085342, \\ \mu(n=3.6) &= 1.86926 93927 72097, \\ (* 0.01761417 * 10.30159^2 &= 1.86926 42743, Horedt *) \\ \mu(n=3.4) &= 1.91319 90558 20976, \\ (* 0.02431955 * 8.869580^2 &= 1.91320 56075, Horedt *) \end{aligned} \tag{55}$$

10.2.9.  $n=4$

$$\begin{aligned} n &= 4, \\ X_0 &= 14.971546348838093, \\ y_0'(X_0) &= -0.008018078806403244, \\ y_0''(X_0) &= 0.0010711089715893653, \\ y_0'''(X_0) &= -0.0002146289260907, \\ y_1(X_0) &= 0.13250008777227387, \\ y_1'(X_0) &= 0.0006870175622715013, \\ y_1''(X_0) &= -0.00009177643327735738, \\ y_2(X_0) &= -0.020971140923665713, \\ y_2'(X_0) &= -0.0002923287696798706, \\ X_1 &= 16.525166560655304, \\ X_2 &= 17.040461450049975, \\ X(n = 4.1) &= 16.7944\,67619, \text{ (* } 16.81377\,38705, \text{ Jabbar *)} \\ X(n = 3.9) &= 13.48943\,43072, \text{ (* } 13.47384\,13948, \text{ Jabbar *)} \\ \mu_0 &= 1.7972299144392503, \\ \mu_1 &= -0.1539930629857884, \\ \mu_2 &= 0.06552467522560379, \\ \mu(n = 4.1) &= 1.78248\,58548\,92927, \\ & \text{ (* } 0.006305183 * 16.81378^2 = 1.78249\,53973, \text{ Horedt *)} \\ \mu(n = 3.9) &= 1.81328\,44674\,90085, \\ & \text{ (* } 0.009988109 * 13.47384^2 = 1.81328\,48994, \text{ Horedt *)}. \end{aligned} \tag{56}$$

10.2.10.  $n=9/2$

$$\begin{aligned} n &= 9/2, \\ X_0 &= 31.836463244695223, \\ y_0'(X_0) &= -0.0017145489124289180377, \\ y_0''(X_0) &= 0.0001077097602991190, \\ y_0'''(X_0) &= -0.000010149660105577, \\ y_1(X_0) &= 0.1171972229781507365218, \\ y_1'(X_0) &= 0.000078867143804445547, \\ y_1''(X_0) &= -4.95451666212243895495976 \cdot 10^{-6}, \\ y_2(X_0) &= -0.01868541777734960517998, \\ y_2'(X_0) &= -0.00008663458543571578, \\ X_1 &= 68.35455210905772991040946, \\ X_2 &= 139.00687095347173451822, \\ X(n=4.6) &= 40.06198\,71651, \text{ (* } 40.41322\,96343, \text{ Jabbar *)} \\ X(n=4.4) &= 26.39107\,67433, \text{ (* } 26.15891\,96485, \text{ Jabbar *)} \\ \mu_0 &= 1.7377988676660324, \\ \mu_1 &= -0.0799366131848989555, \\ \mu_2 &= 0.087809384368989, \\ \mu(n=4.6) &= 1.7306833001912325, \\ (* 0.001059709 * 40.41324^2 &= 1.73074\,84954, \text{ Horedt *)}, \\ \mu(n=4.4) &= 1.7466706228282123, \\ (* 0.002552517 * 26.15892^2 &= 1.74665\,95493, \text{ Horedt *)}. \end{aligned} \tag{57}$$

## 11. Summary

In this paper we present the exact numerical solutions for the internal structure and global parameters of polytropic models of arbitrary index  $0 < n < 5$ . The perturbation method used here is not rigorously founded by means of the theory of differential equations and there are some problems about application of the method in the interval of argument where the perturbation function is of the same order or even larger than the initial non-perturbed function. The problem of justification is here similar to the problem of rotationally distorted polytropes which was already discussed in the astrophysical literature (see e.g. Chandrasekhar & Lebovitz (1962)). Still validity of any method in applicational sciences (as astrophysics is such relative to mathematics) may be checked with numerical calculations and (astro)-physical "common sense", and we showed in this paper that the perturbation method is applicable to the problem of the structure and the global parameters of the polytropic models.

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